

CSCI 7000 Fall 2023: Problems on Fibonacci Series

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To work on during class:

The Fibonacci sequence is defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$; its first few terms are thus 0, 1, 1, 2, 3, 5, 8, 13.

1. Prove that $F_n \leq 2^n$ for all n in as many different ways as possible, including an inductive proof.
2. Prove that $F_n \gtrsim c(\sqrt{2})^n$ for some $c > 0$ and for all $n \geq 1$ in as many different ways as possible, including an inductive proof.
3. Try to generalize your proofs in (1) to get tighter upper bounds; try to generalize your proofs in (2) to get tighter lower bounds. Can you find the limit of these methods?
4. For all $n \geq 0$, we have

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

Use the eigenvalues of this matrix to determine the asymptotic growth rate of the Fibonacci sequence.